Algorithmic Thinking

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Matrices

Matrices play an important role in graph theory. In particular, the connectivity of a graph can be represented by an *adjacency matrix*. So, what is a matrix?

A *matrix* is a rectangular array of numbers. If the matrix has m rows and n columns, we say it is an $m \times n$ matrix. The following is an example of a 4×3 matrix A:

$$A = \begin{bmatrix} 5 & 10 & -1 \\ 12 & 2 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 100 \end{bmatrix} \tag{1}$$

We use the convention that rows are numbered $0 \ldots m-1$ and columns are numbered $0 \ldots n-1$. We use the notation $A[i,j]$ to denote the value of the entry at row i and column j. For example, for the matrix A above, we have $A[0,0]5$.

A square matrix is a matrix that has the same number of rows and columns. An $n \times n$ (square) matrix is a matrix that has n rows and n columns.

Two important operations on matrices are *addition* and *multiplication*. To add two matrices A and B, they must have the same number of rows and the same number of columns. If A and B are two $m \times n$ matrices, then $C = A + B$ is a matrix that satisfies

$$C[i,j] = A[i,j] + B[i,j]$$

for every $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

To multiply two matrices A and B (denoted $A \times B$), the number of columns in A must be equal to the number of rows in B (the number of rows in A and the number of columns in B can be of any value). Let A be an $m \times k$ matrix and B be a $k \times n$ matrix. Then, the product of the two matrices, $C = A \times B$ satisfies

$$C[i,j] = \sum_{\ell=0}^{k-1} \left(A[i,\ell] \cdot B[\ell,j]\right)$$

for every $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

Graphs and matrices. Given an undirected graph $g = (V,E)$, where $V = \{0, \ldots, n-1\}$, the graph can be represented by an *adjacency matrix* A_g (or, A), where

$$A[i,j] = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Notice that for an undirected graph, the adjacency matrix has to be *symmetric*; that is, for every i and j, $A[i,j] = A[j,i]$.

For a directed graph $g = (V,E)$ with set $V = \{0, \ldots, n-1\}$ of nodes, the adjacency matrix does not have to be symmetric, since the presence of an edge (i,j) does not mean the edge (j,i) have to be in the graph.

It is important to note that adjacency matrices of (unweighted) directed and undirected graph are binary; that is, every entry in the adjacency matrix is either 0 or 1 to denote the absence and presence, respectively, of an edge.